Exact solutions of a generalized nonlinear Schrödinger equation

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(Received 5 May 2008; published 1 August 2008)

Exact chirped soliton solutions of a generalized nonlinear Schrödinger equation with the cubic-quintic nonlinearities as well as the self-steeping were obtained using a variable parametric method. It was found that the formation of solutions is determined by the sign of a joint parameter solely. By performing numerical simulations, the chirped solutions are stable under perturbations.

DOI: 10.1103/PhysRevE.78.026602 PACS number(s): 42.65.Wi

The nonlinear Schrödinger (NLS) equation is widely applied for the study of solitons in nonlinear optics and plasma physics as well as nonlinear dispersive water waves [1,2]. Theoretically, there exist many methods for solving the NLS equation, such as Darboux-Bäcklun transform (DBT), inverse scattering transform (IST) [2], self-similarity technique [3], and variational approach [4,5]. In general, DBT and IST can obtain N-soliton solution to the NLS equation through a long procedure. Self-similarity technique is broadly applicable for finding self-similar solutions to a range of nonlinear differential equations. Variational approach is a useful approximation which generally solves a wide class of nonlinear equations.

In this paper, a variable parametric method is adopted for solving a generalized nonlinear Schrödinger (GNLS) equation with the cubic-quintic nonlinearities as well as the self-steeping. We will show that there exist exact chirped solutions, controlled by the sign of the joint parameter $\Omega=\sigma^2+8\delta k$. Employing the Drude model, we verify numerically the stability of the chirped solutions.

The NLS equation is widely used for descriptions to the propagation of picosecond pulses in the literature [1]. It has been demonstrated that when the pulses are shorter than 100 femtoseconds, the higher order effects in nonlinear media become important, and therefore the governing equation should still include third-order dispersion (TOD), the self-steepening, and the self-frequency shift [6]. It is interesting that the propagation of ultrashort pulses at least a few tens of optical cycles in duration can be described by the GNLS equation [7]

$$i\psi_\xi+\frac{k}{2}\psi_{\tau\tau}+i\sigma(|\psi|^2\psi)_\tau+\rho|\psi|^2\psi+3\delta|\psi|^4\psi=0, \qquad (1)$$

where $\psi(\xi,\tau)$ is the complex envelope of the electric field, ξ and τ are the retarded coordinates, and k, σ, ρ, and δ are the real parameters related to group velocity dispersion (GVD), self-steepening, cubic nonlinearity, and quintic nonlinearity [1]. Recently, the model (1) is used to characterize wave propagation in a negative index material (NIM). In this situation, the sign of GVD can be positive or negative and self-steepening characterizes the front of the pulse, different from the case of ordinary materials [7].

In the case when $\sigma=\delta=0$, Eq. (1) can be reduced into the basic NLS equation, including the GVD and the cubic nonlinearity [1]. It has been well-known that this NLS equation admits bright or dark soliton-type pulse propagation. For $\delta=0$, it becomes the derivative NLS (DNLS) equation governing the propagation of NLS soliton in the presence of Kerr dispersion [8]. When $\delta\neq 0$, Eq. (1) cannot pass the Painléve PDE test [9], and cannot be solved by employing DBT or IST. To the best of our knowledge, exact analytic solutions to model (1) have been absent. In this work, we find exact analytic solutions to Eq. (1) using a method of variable parameters. The main steps are as follows: First, we choose the soliton solution to the basic NLS equation as a trial solution, where part parameters are the functions of the longitudinal and transverse coordinates. Second, we substitute the trial solution into the equation considered and find these parametric functions.

According to the standard solution to the basic NLS equation [1], the bright solution to Eq. (1) in the form takes the compact expression

$$\psi(\xi,\tau)=\eta(\xi,\tau)\mathrm{sech}(\kappa\tau+\lambda\xi)e^{i\Phi(\xi,\tau)}, \qquad (2)$$

where κ and λ are real parameters, and the amplitude $\eta(\xi,\tau)$ and the phase $\Phi(\xi,\tau)$ are the real variable functions of the retarded coordinates ξ and τ. The phase shift $\Phi(\xi,\tau)$ reads

$$\Phi(\xi,\tau)=E\tau+F\xi+\phi(\xi,\tau), \qquad (3)$$

where E and F are real parameters. For the basic NLS equation, the phase shift ϕ is a constant parameter. However, when there are higher order terms in the NLS equation, $\phi(\xi,\tau)$ is generally a nonlinear function of the retarded coordinates ξ and τ. Substituting Eqs. (2) and (3) into Eq. (1), requiring that the real and imaginary parts of each term be separately equal to zero, and considering that Eq. (2) for arbitrary ξ and τ satisfies Eq. (1), we obtain the following system of equations:

$$(1-q^2)\left(\frac{1}{2}k\kappa\phi_{qq}+k\kappa\phi_q\frac{\eta_q}{\eta}+3\sigma\eta\eta_q\right)-q(2k\kappa\phi_q+3\sigma\eta^2)=0, \qquad (4)$$

$$(1-q^2)\left(\frac{1}{2}k\kappa^2\frac{\eta_{qq}}{\eta}-\frac{1}{2}k\kappa^2\phi_q^2-\sigma\kappa\eta^2\phi_q+3\delta\eta^4\right)-(\rho-\sigma E)\eta^2-2k\kappa^2 q\frac{\eta_q}{\eta}-k\kappa^2=0, \quad (5)$$

where $q=\tanh(\kappa\tau+\lambda\xi)$, and there exist the relations

$$\lambda+k\kappa E=0, \quad 2F-k\kappa^2+kE^2=0. \quad (6)$$

It should be stressed that Eqs. (4) and (5) are a set of coupled differential equations. Owing to an extended variable separation method in terms of the ratio $q/(1-q^2)$, we can split Eq. (4) into two equations in the form

$$\frac{1}{2}k\kappa\phi_{qq}+k\kappa\phi_q\frac{\eta_q}{\eta}+3\sigma\eta\eta_q=qf(q), \quad (7)$$

$$2k\kappa\phi_q+3\sigma\eta^2=(1-q^2)f(q), \quad (8)$$

where $f(q)$ is real. Combining Eqs. (7) and (8), we easily find the expression, $f(q)=D\eta^{-2}(1-q^2)^{-3}$. Here, D is an integral constant. Considering Eqs. (4) and (8), one has $2k\kappa\phi_q+3\sigma\eta^2=D\eta^{-2}(1-q^2)^{-2}$. When $q\to\pm 1$, $D=0$ must hold. By applying this result $f(q)\equiv 0$, the substitution of Eq. (8) into Eq. (5) yields

$$4k^2\kappa^2(1-q^2)\frac{\eta_{qq}}{\eta}-16k^2\kappa^2 q\frac{\eta_q}{\eta}+3\Omega(1-q^2)\eta^4+8k\Delta\eta^2-8k^2\kappa^2=0, \quad (9)$$

where $\Omega=\sigma^2+8\delta k$ and $\Delta=\rho-\sigma E$. To look for the solution to Eq. (9), we introduce the transformation

$$\eta(q)=\frac{1}{\sqrt{1+cq^2}}g, \quad (10)$$

where $c>-1$ and g are real. If g is the function of q, g can be expanded into a Maclaurin series. Substituting Maclaurin series for g into Eq. (10), we reexpress Eq. (9) and make the coefficient of each power of q equal to zero. As a result, g can only be constant and satisfies

$$g^2=-\frac{k}{\Delta}\kappa^2(c-1), \quad (11)$$

$$\Omega\kappa^2(c-1)^2-4c\Delta^2=0. \quad (12)$$

Apparently, the sign of the constant c is the same as the joint parameter Ω, relating to δ, k, and σ. Now, we discuss three cases separately.

(i) $\Omega=\sigma^2+8\delta k>0$. In Eq. (12), the parameter c is positive. Using Eqs. (11) and (12), we find

$$\Delta=-k(B^2-A^2)k\sqrt{\Omega}, \quad g=-2Bk\Omega^{-1/4}. \quad (13)$$

Here, $c=B^2/A^2$ and $\kappa=-2ABk$. The real A and B are arbitrary. By combining Eqs. (13) and (10) with (8), the soliton solution to Eq. (1) is given by

$$\psi_+(\xi,\tau)=\frac{\kappa\Omega^{-1/4}\,\mathrm{sech}(\kappa\tau+\lambda\xi)}{\sqrt{A^2+B^2q^2}}\times\exp\left\{i\left[E\tau+F\xi+\frac{3\sigma}{\sqrt{\Omega}}\tan^{-1}\left(\frac{B}{A}q\right)+\phi_0\right]\right\}, \quad (14)$$

where $E=[\rho+(B^2-A^2)k\sqrt{\Omega}]/\sigma$, $\lambda=2ABk^2E$, and $F=k(\kappa^2-E^2)/2$. It is pointed out that for $\delta=0$, the solution of Eq. (14) gives the same result in Ref. [7]. The resultant chirp, including linear and nonlinear contributions, can be obtained readily

$$\delta w_+(\tau)=-E+\frac{6A^2B^2k\sigma\,\mathrm{sech}^2(\kappa\tau+\lambda\xi)}{\sqrt{\Omega}[A^2+B^2\tanh^2(\kappa\tau+\lambda\xi)]}. \quad (15)$$

(ii) $\Omega=\sigma^2+8\delta k<0$, i.e., $-1<c<0$. In this sense, the use of $c=-B^2/A^2>-1$ results in the second solution to Eq. (1) and the associated chirp

$$\psi_-(\xi,\tau)=\frac{\kappa|\Omega|^{-1/4}\,\mathrm{sech}(\kappa\tau+\lambda\xi)}{\sqrt{A^2-B^2q^2}}\times\exp\left\{i\left[E\tau+F\xi+\frac{3\sigma}{\sqrt{|\Omega|}}\tanh^{-1}\left(\frac{B}{A}q\right)+\phi_0\right]\right\}, \quad (16)$$

$$\delta w_-(\tau)=-E+\frac{6A^2B^2k\sigma\,\mathrm{sech}^2(\kappa\tau+\lambda\xi)}{\sqrt{|\Omega|}[A^2-B^2\tanh^2(\kappa\tau+\lambda\xi)]}, \quad (17)$$

where $E=[\rho-(B^2+A^2)k\sqrt{|\Omega|}]/\sigma$, and the others are the same as those in the case (i).

(iii) $\Omega=\sigma^2+8\delta k=0$, i.e., $c=0$. From Eqs. (8)–(10), under the condition that $(\rho-\sigma E)k>0$ there also is a soliton solution to Eq. (1) as follows:

$$\psi_0(\xi,\tau)=\kappa\sqrt{\frac{k}{\rho-\sigma E}}\,\mathrm{sech}(\kappa\tau+\lambda\xi)\times\exp\left[i\left(E\tau+F\xi-\frac{3\sigma\kappa}{2(\rho-\sigma E)}q+\phi_0\right)\right], \quad (18)$$

where $E=-k\kappa/\lambda$ and $F=k\kappa^2(\lambda^2-k^2)/2\lambda^2$ are dependent on free parameters κ and λ. The chirp takes an expression

$$\delta w_0(\tau)=-E+\frac{3\sigma\kappa^2}{2(\rho-\sigma E)}\mathrm{sech}^2(\kappa\tau+\lambda\xi). \quad (19)$$

We note that in the chirped expressions, Eqs. (15), (17), and (19), the nonlinear chirps are determined by the cubic or quintic nonlinearity and pause self-steepening.

Next, we apply model (1) to NIMs so as to illustrate stability of solition solutions (14), (16), and (18). For the propagation of ultrashort pulses in NIMs, the GVD, self-steepening, cubic nonlinearity, and quintic nonlinearity are expressed as $k=\frac{1}{\beta n}(\frac{1}{V_g^2}-\alpha\gamma-\beta\frac{\epsilon\gamma'+\mu\alpha'}{4\pi})$, $\sigma=-\frac{\chi^{(3)}}{2V_g n^2}[\mu-V_g n(\gamma+\mu)]$, $\rho=\frac{1}{2n}\beta\mu\chi^{(3)}$, and $\delta=-\frac{1}{24n^3}\beta(\mu\chi^{(3)})^2$, respec-

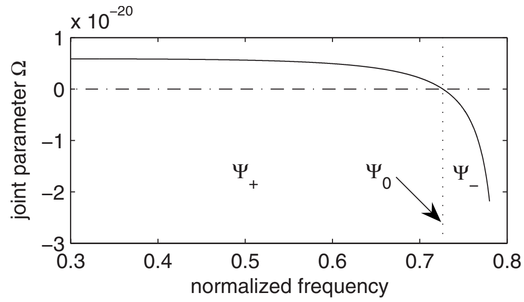

FIG. 1. Control parameter Ω vs normalized frequency $\tilde{\omega}$ for a NIM with $\Gamma=0$ and $\chi^{(3)}=10^{-10}$.

tively [7]. Here $V_g=\frac{2n}{\epsilon\gamma+\mu\alpha}$, $\beta=2\pi\tilde{\omega}$, $\alpha=\frac{\partial[\tilde{\omega}\epsilon(\tilde{\omega})]}{\partial\tilde{\omega}}$, $\alpha'=\frac{\partial^2[\tilde{\omega}\epsilon(\tilde{\omega})]}{\partial\tilde{\omega}^2}$, $\gamma=\frac{\partial[\tilde{\omega}\mu(\tilde{\omega})]}{\partial\tilde{\omega}}$, and $\gamma'=\frac{\partial^2[\tilde{\omega}\mu(\tilde{\omega})]}{\partial\tilde{\omega}^2}$, where ϵ, μ, n, $\chi^{(3)}$, and $\tilde{\omega}$ denote dielectric susceptibility, effective magnetic permeability, index of refraction, the coefficient of cubic nonlinearity, and normalized frequency, respectively. It is particularly stressed that one must be careful about applying the solution (16) to research nonlinear pulse propagation in NIMs, since the range of validity of Eq. (1) cannot be extended to regions where $n\to 0$ [7,10].

The issue of stability of solitary solutions is an interesting and complex topic. In general, bright soliton solutions to the one-dimensional cubic-quintic nonlinear Schrödinger equation are known to be stable [11]. Since Eq. (1) contains such cubic-quintic nonlinearities, as well as a derivative nonlinearity (i.e., self-steepening), it is reasonable to conjecture that the analytic solutions (14), (16), and (18) should be stable. To confirm the stability of the soliton solutions by directly solving Eq. (1) with the split-step Fourier method [12], we cite a Drude model described by $\epsilon(\tilde{\omega})=1-\tilde{\omega}^{-2}$ and $\mu(\tilde{\omega})=1-0.64\tilde{\omega}^{-2}$ with negligible absorption [13]. Figure 1 shows the control parameter Ω, the normalized frequency $\tilde{\omega}$, and ranges of the solutions for $\chi^{(3)}=10^{-10}$. The normalized frequency $\tilde{\omega}_0=0.7263$ is found for $\Omega=0$, corresponding to the solution (18). There exist the solutions $\psi_+(\xi,\tau)$ and $\psi_-(\xi,\tau)$ in the regimes of $\tilde{\omega}<\tilde{\omega}_0$ and $\tilde{\omega}>\tilde{\omega}_0$, respectively. Based on this concrete model, when $\tilde{\omega}_0=0.7$ and $\chi^{(3)}=10^{-10}$, the typical values of the parameters in Eq. (1) are as follows: $k=0.1168$, $\sigma=-3.5411$, $\rho=1.1926$, and $\delta=9.5489\times 10^{-22}$. Figure 2 shows the amplitude of the pulse for $A=1$, $B=0.4$, and $\phi_0=0$ in the case of $\Omega>0$. The insets display the comparison of the analytical results (14) and (15) with the numerical simulations. For convenience, we have normalized the solution satisfying $\psi\to\Omega^{1/4}\psi/(2Bk)$. Meanwhile, time is normalized by $2B(\rho^2k^2/\Omega)^{1/4}$, too. It is clear that our analytical results (14) and (15) are in excellent agreement with the numerical simulations, as shown in the insets. Furthermore, we also look for the evolution of the initial pulse by adding Gaussian white noise and by evolving from an initial chirped Gaussian pulse for multiplication of

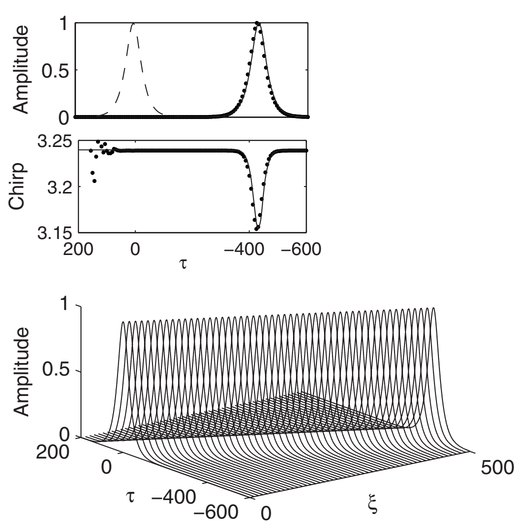

FIG. 2. Evolution of an initial pulse described as the exact solutions (14) and (15) for $A=1$, $B=0.4$, $\phi_0=0$, and $\tilde{\omega}=0.7$ in the regime of $\Omega>0$. We have normalized the solution and time satisfying $\psi\to\Omega^{1/4}\psi/(2Bk)$ and $2B(\rho^2k^2/\Omega)^{1/4}$, respectively. Insets show the comparison of our analytical results (15) and (16) at $z=500$ (solid line) with the numerical simulations (circle) and the initial profile (dotted line).

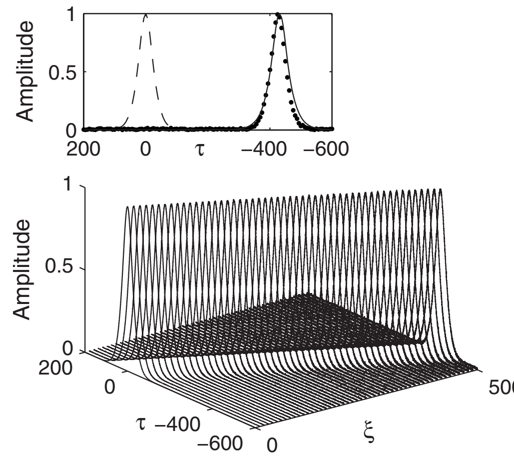

FIG. 3. Evolution of an initial pulse that is the same as in Fig. 2 except for considering the multiplicative white noise with the strength $\gamma_1=10^{-3}$ and the additive white noise strength $\gamma_2=0.1$. We normalize the solution and time as in Fig. 2.

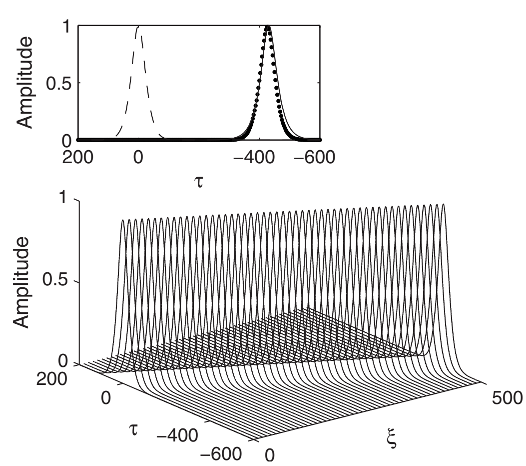

FIG. 4. Evolution of an initial pulse that is the same as in Fig. 1 except for multiplication of the exact solution by $\exp[-0.1(\kappa\tau)^2]$. Here, the solution and time are normalized, too.

the exact solution by $\exp[-0.1(\kappa\tau)^2]$ shown in Figs. 3 and 4. By performing numerical simulations, we demonstrate the stability of those nonlinearly chirped pulses, corresponding to the analytical results (16)–(19). From the evolution of waves, we find that the chirped solution and the background, are quite stable under finite perturbations.

In conclusion, exact chirped soliton solutions of a generalized nonlinear Schrödinger equation with cubic-quintic nonlinearities as well as self-steeping have been found analytically, owing to a variable parametric method. The analytic results showed that the formation of solutions can be determined by the sign of the joint parameter $\Omega=\sigma^2+8\delta k$. In NIMs, the sign of the GVD and k can be positive or negative, thereby the present solutions could exist. In the Drude approximation, numerical simulations indicated that the chirped solutions are quite stable under finite perturbations.

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